Algebraic semantics of logic programs with negation: Characterization of weak interlaced bilattices

Michiro Kondo
School of Information Environment
Tokyo Denki University, Inzai, 270-1382
JAPAN
kondo@sie.dendai.ac.jp

Abstract
We consider fundamental properties of weak interlaced bilattices and give a characterization theorem of them, that is, for any weak interlaced bilattice $W$ there exists a bounded lattice $L$ such that $W$ can be embedded into a typical weak interlaced bilattice $K(L)$. Hence, any interlaced bilattice can be embedded into the weak interlaced bilattice $K(L)$ for some lattice $L$.

1 Introduction
It is well-known that the Kleene’s 3-valued logic plays an important role in the field of multiple-valued logics. The logic has three values false, true, and $\perp$ (unknown) as truth values. These values have two informal orderings concerning "amount of knowledge" and "degree of truth". For example, if a certain proposition such as Goldbach’s conjecture which is not known true of false is considered, then it is possible that the truth value of the proposition can be determined as true or false with increasing knowledge. Thus in the ordering of knowledge, the unknown symbol $\perp$ is smaller than true and false. A sentence with $\perp$ is between false and true in the ordering of degree of truth. In this way it can be considered that the three valued logic has two orderings. Belnap ([2]), Ginsberg([5]), and others proposed a concept of a bilattice which has two orderings and proved some fundamental results ([1, 3, 4]). It is shown by Fitting ([3]) that bilattices can give a uniform semantics for many languages of logic programming. Since then the theory of bilattices is a hot research field.

In the semantic theory of logic programs, bilattices play a very important role. Let $P$ be a program, that is, it is a finite sequence of negation-free formulas and $\Pi$ the set of all negation-free formulas. For a bounded lattice $L$, a map $v: \Pi \rightarrow L$ is called a valuation on $L$. For all valuations $v$ and $v'$ on $L$, we define a order relation $\leq$ as follows. For each formula $A \in \Pi$,

$$v \leq v' \iff v(A) \leq v'(A).$$

It is well-known that

Proposition 1. The set of all valuations on a bounded lattice $L$ forms a complete lattice with

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respect to the order ≤. Thus, there is a fixed point for any monotone valuation.

For every program \( P \), we define a map \( \Phi_P \) between valuations as follows:

\[
\Phi_P(v)(A) = \begin{cases} 
v(A) & \text{if } A \in P \\
v(B) & \text{if } A \leftarrow B \in P \\
0 & \text{otherwise}
\end{cases}
\]

Then we have

\[ v \leq v' \implies \Phi_P(v) \leq \Phi_P(v') \]

hence we define a semantics of program \( P \) by the least fixed point of \( \Phi_P \). Extending the semantics to the case of arbitrary programs with negation formulas, there is a serious problem in the semantics. In general case we cannot guarantee the existence of the fixed points of the lattice of valuations, because the order on the set of valuations is not monotone. To overcome this difficulty, Fitting ([3]) and Ginsberg ([5]) proposed the theory of bilattices.

On the other hand, in the theory of Fuzzy logics, each proposition has a closed interval \([a, b]\) as a truth value. By transferring the support set \( R \) of all real numbers to any lattice, this can be extended to a general case. Let \( L \) be a lattice and \( K(L) \) be the set of all closed intervals of \( L \). In this case two natural orderings are defined on \( K(L) \).

For \([a, b], [c, d] \in K(L)\), if \([a, b] \subseteq [c, d]\) then the knowledge in \([a, b]\) is greater than that in \([c, d]\). Thus we set \([c, d] \sqsubseteq [a, b] \) if \([a, b] \subseteq [c, d]\). Likewise we also define \([a, b] \sqsubseteq_k [c, d]\) if \(a \leq c\) and \(b \leq d\), because \([c, d]\) is greater than \([a, b]\) in the ordering degree of truth. The structure \( K(L) = \langle K(L), \sqsubseteq_t, \sqsubseteq_k \rangle \) which precise definition is given below has the property of weak interlaced bilattice.

In [3, 4], Fitting, Font and Moussavi have investigated the structure of \( K(L) \) and proved that if \( L \) is a bounded lattice, then \( K(L) \) is a weak interlaced bilattice ([4]). Now does the converse hold?, that is, is there a lattice \( L \) such that \( \mathcal{W} \cong K(L) \) for every weak interlaced bilattice \( \mathcal{W} \)?

Clearly we answer "No". Because we have a simple counter-example. Let \( \mathcal{B} \) be a set \([0, p, \bot, q, 1]\) with \(0 \leq_t p \leq_t \bot \leq_t q \leq_t 1\), \(\bot \leq_k p \leq_k 0\) and \(\bot \leq_k q \leq_k 1\). It is obvious that \( \mathcal{B} \) is a weak interlaced bilattice. Suppose that there is a lattice \( L \) such that \( \mathcal{B} \cong K(L) \).

If \(|L| \geq 3\), then there exists an element \( a \in L \) such that \(0 < a < 1\). For that element we have \([0, a], [0, 1], [0, 1], [a, a], [1, 1] \in K(L)\) and \(|K(L)| \geq 6\). Since \(|\mathcal{B}| = 5\), it must be \(|L| \leq 2\). But, in this case, we have \(|K(L)| \leq 3\). This means that there is no lattice \( L \) such that \( \mathcal{B} \cong K(L) \).

Now we settle a more general question.

**Question**: For every weak interlaced bilattice \( \mathcal{W} \), is there a lattice \( L \) such that \( \mathcal{W} \) can be embedded to \( K(L) \)?

In this paper we study properties of \( K(L) \) and answer the question.

## 2 Definition of \( K(L) \)

We define a structure \( K(L) \) for any lattice \( L \). Let \( L = (L, \leq) \) be a lattice and \( K(L) \) be the set of all closed intervals of \( L \), that is,

\[
K(L) = \{ [a, b] | a \leq b, a, b \in L \}
\]

\[
[a, b] = \{ x \in L | a \leq x \leq b \}.
\]

For any \([a, b], [c, d] \in K(L)\), we define two orderings \( \sqsubseteq_t, \sqsubseteq_k \) on \( K(L) \) as follows:

\[
[a, b] \sqsubseteq_t [c, d] \iff a \leq c, b \leq d
\]

\[
[a, b] \sqsubseteq_k [c, d] \iff a \leq c, b \geq d
\]
We set $\mathcal{K}(L) = < K(L), \leq_t, \leq_k >$. It is obvious from definition that $[0, 0] \cup [1, 1]$ is the minimum (maximum) element with respect to $\leq_t$. On the other hand, while $[0, 1]$ is the minimum element, there is no maximum element with respect to the ordering $\leq_k$. This means that $\mathcal{K}(L)$ is a lattice with respect to $\leq_t$ and is a semi-lattice concerning $\leq_k$.

Next we give definitions of an interlaced bilattice and of a weak interlaced bilattice. A relational system $< B, \leq_t, \leq_k >$ is called an interlaced bilattice if it satisfies

1. $B$ is a non-empty set

2. $< B, \leq_t >$, $< B, \leq_k >$ are bounded lattices and satisfy

   \[(a) \ x \leq_t y \Rightarrow x \otimes z \leq_t y \otimes z, \ x \oplus z \leq_t y \oplus z \]

   \[(b) \ x \leq_k y \Rightarrow x \land z \leq_k y \land z, \ x \lor z \leq_k y \lor z \]

where four operators are defined by

\[
\begin{align*}
\inf_{\leq_t} \{ x, y \} &= x \land y \\
\sup_{\leq_t} \{ x, y \} &= x \lor y \\
\inf_{\leq_k} \{ x, y \} &= x \otimes y \\
\sup_{\leq_k} \{ x, y \} &= x \oplus y
\end{align*}
\]

By 0(1), we mean the minimum (maximum) element with respect to the ordering $\leq_t$. We also denote by $\bot (\top)$ the minimum (maximum) element concerning to $\leq_k$.

A map $\neg$ from $B$ into itself is called a negation if

\[
\begin{align*}
x \leq_t y &\Rightarrow \neg y \leq_t \neg x \\
x \leq_k y &\Rightarrow \neg x \leq_k \neg y \\
\neg \neg x &= x.
\end{align*}
\]

For lattices $L_1 = < L_1, \land_1, \lor_1 >$ and $L_2 = < L_2, \land_2, \lor_2 >$, we define operations $\land, \lor, \otimes, \oplus$ on the product $L_1 \times L_2$.

\[
\begin{align*}
(a, b) \land (c, d) &= (a \land_1 c, b \lor_2 d) \\
(a, b) \lor (c, d) &= (a \lor_1 c, b \land_2 d) \\
(a, b) \otimes (c, d) &= (a \land_1 c, b \land_2 d) \\
(a, b) \oplus (c, d) &= (a \lor_1 c, b \lor_2 d).
\end{align*}
\]

The structure $L_1 \otimes L_2 = < L_1 \times L_2, \land, \lor, \otimes, \oplus >$ is called a Ginsberg product. There are some fundamental results about the structure:

**Proposition 2 (Fitting).** If $L_1, L_2$ are bounded lattices then the Ginsberg product $L_1 \otimes L_2 = < L_1 \times L_2, \land, \lor, \otimes, \oplus >$ is an interlaced bilattice. Especially, $L \circ L$ is an interlaced bilattice with negation $\neg$, where $\neg$ is defined by $\neg (a, b) = (b, a)$.

It is proved that the converse holds by Avron ([1]).

**Proposition 3 (Avron).** For any interlaced bilattice $\mathcal{B}$, there are bounded lattices $L_1, L_2$ such that $\mathcal{B} \cong L_1 \circ L_2$. In particular, for any interlaced bilattice $\mathcal{B}$ with negation, there is a bounded lattice $L$ such that $\mathcal{B} \cong L \circ L$.

It is clear from definition that orderings $\sqsubseteq_t$, $\sqsubseteq_k$ on $\mathcal{K}(L)$ are the same as $\leq_t$, $\leq_k$ on Ginsberg product $L \circ L$, respectively:

\[
\begin{align*}
\sqsubseteq_t \text{ in } \mathcal{K}(L) &\iff \leq_t \text{ in } L \circ L \\
\sqsubseteq_k \text{ in } \mathcal{K}(L) &\iff \leq_k \text{ in } L \circ L
\end{align*}
\]

Hence in the following we use the same symbols $\land, \lor, \otimes, \oplus$ in $\mathcal{K}(L)$ and in $L \circ L$.

Next we give a definition of a weak interlaced bilattice according to Font ([4]). A structure $W = < W, \leq_t, \leq_k >$ is called a weak interlaced bilattice if

\[
\begin{align*}
\neg \neg x &= x.
\end{align*}
\]
1. \( < W, \leq_l > \) : lattice

2. \( < W, \leq_k > \) : meet semilattice

3. \( a \leq_k b, c \leq_k d \Rightarrow a \land c \leq_k b \lor d, a \lor c \leq_k b \land d \)

4. \( a \leq_l b, c \leq_l d \Rightarrow a \lor c \leq_l b \land d, \)

5. \( a \leq_l b, c \leq_l d \Rightarrow a \lor c \leq_l b \land d \) if \( a \lor c \) and \( b \land d \) exist.

3 Properties of weak interlaced bilattices

For any weak interlaced bilattice \( W \), if we define

\[
L_1 = \{ x \in W \mid x \leq_k 0 \} = [\bot, 0]_k \\
L_2 = \{ x \in W \mid x \leq_k 1 \} = [\bot, 1]_k
\]

then we have

Proposition 4.

\[
L_1 = [\bot, 0]_k = [0, \bot]_l \\
L_2 = [\bot, 1]_k = [\bot, 1]_l
\]

Proof. Let \( x \in [\bot, 0]_k \). Since \( \bot \leq_k x \leq_k 0 \), we have \( \bot \lor \bot \leq_k x \lor \bot \leq_k 0 \lor \bot \) by definition of weak interlaced bilattice. From \( \bot \lor \bot = 0 \lor \bot = \bot \), it follows that \( x \lor \bot = \bot \) and hence that \( x \leq_l \bot \). This means \([\bot, 0]_k \subseteq [0, \bot]_l \).

Conversely, suppose \( x \in [0, \bot]_l \). If we put \( u = 0 \lor x \), then it is clear that \( u \leq_k 0 \) and \( u \leq_k x \). Since \( 0 \leq_l x \), we have \( 0 \lor x \leq_l x \lor x = x \) and hence \( u \leq_l x \). It follows from \( \bot \leq_k u \) that \( x \lor \bot \leq_k x \lor u \). Since \( x \leq_l \bot \), we also have \( x \lor \bot = x \). On the other hand, since \( u \leq_l x \), we get \( u \lor x = u \). Theses imply that \( x \leq_k u \) and hence that \( x = u \). Thus we have \( x \leq_k 0 \).

Namely, we have \([0, \bot]_l \subseteq [\bot, 0]_k \).

The second equation can be proved similarly. \( \square \)

The result implies that \( L_1 \) and \( L_2 \) are lattices with ordering \( \leq_1 \) and \( \leq_2 \) in \( B \), respectively, where \( \leq_1 \) and \( \leq_2 \) are defined by

\[
\leq_1 = \leq_l = \leq_k \\
\leq_2 = \leq_l = \leq_k
\]

Thus we can consider the Ginsberg product \( L_1 \odot L_2 \), which becomes an interlaced bilattice. Moreover we can prove

Proposition 5. Let \( W \) be any weak interlaced bilattice. For any \( x \in W \), we have

\[
x = (x \otimes 0) \oplus (x \otimes 1) = (x \land \bot) \lor (x \lor \bot)
\]

Proof. See Avron [1] Cor.3.8 \( \square \)

Now we investigate a relation between a weak interlaced bilattice \( W \) and an interlaced bilattice \( L_1 \odot L_2 \) constructed from \( W \).

Lemma 1. A map \( \xi : W \rightarrow L_1 \times L_2 \) defined by \( \xi(x) = (x \otimes 1, x \otimes 0) = (x \lor \bot, x \land \bot) \) is an embedding.

This means that

Theorem 1. Any weak interlaced bilattice can be embedded into an interlaced bilattice.

4 Answer to the question

In this section we give a positive answer to the question in the introduction. Since any weak interlaced bilattice \( W \) can be embedded to an interlaced bilattice, it suffices to show that any interlaced bilattice of a form \( L_1 \odot L_2 \) is embeddable into a weak interlaced bilattice \( K(L) \) for some lattice \( L \). Because, from proposition 3, every interlaced bilattice has a form of \( L_1 \odot L_2 \) for some lattices \( L_1, L_2 \). Let \( L_1 \odot L_2 \) be any interlaced bilattice and \( L \) be a set \( (L_1 \times \{0\}) \cup (L_2 \times \{1\}) \).

We define an order \( \sqsubseteq \) on \( L \). For any element \((a, i), (b, j) \in L \), we define
\[(a, i) \subseteq (b, j) \iff i < j \text{ or } i = j \text{ and } a \leq b\]

It is easy to show that the relation \(\subseteq\) is a partially order on \(L\) and that
\[
(a, i) \wedge (b, j) = \inf\{(a, i), (b, j)\}
\]
\[
= \begin{cases} 
(a \wedge b, i) & \text{if } i = j \\
(a, i) & \text{if } i < j \\
(b, j) & \text{if } i > j
\end{cases}
\]
\[(a, i) \vee (b, j) = \sup\{(a, i), (b, j)\}
\]
\[
= \begin{cases} 
(a \vee b, i) & \text{if } i = j \\
(a, i) & \text{if } i < j \\
(b, j) & \text{if } i > j
\end{cases}
\]

Hence \(L\) is a lattice with this order. Let \(K(L)\) be the set of all elements \([((a, i), (b, j)]\) such that \((a, i) \subseteq (b, j)\) for \((a, i), (b, j) \in L\). In this case, four operators \(\wedge, \vee, \otimes, \oplus\) on \(K(L)\) are defined as follows:

\[
[(a, i), (b, j)] \wedge [(a', i'), (b', j')]
= [(a, i) \wedge (a', i'), (b, j) \wedge (b', j')]
\]
\[
[(a, i), (b, j)] \vee [(a', i'), (b', j')]
= [(a, i) \vee (a', i'), (b, j) \vee (b', j')]
\]
\[
[(a, i), (b, j)] \otimes [(a', i'), (b', j')]
= [(a, i) \wedge (a', i'), (b, j) \wedge (b', j')]
\]
\[
[(a, i), (b, j)] \oplus [(a', i'), (b', j')]
= [(a, i) \vee (a', i'), (b, j) \vee (b', j')]
\]

Of course, the last equation is defined when the inequality \((a, i) \vee (a', i') \leq (b, j) \wedge (b', j')\) is satisfied.

Now we define a map \(\xi : L_1 \odot L_2 \rightarrow K(L)\) by
\[
\xi(a, b) = [(a, 0), (b, 1)]
\]

It is obvious that \(\xi\) is well-defined and injective. We only think of two cases. For the case of \((a, b) \wedge (a', b')\), we have
\[
\xi((a, b) \wedge (a', b'))
= \xi(a \wedge a', b \lor b')
= [(a \wedge a', 0), (b \lor b', 1)]
= [(a, 0) \wedge (a', 0), (b, 1) \lor (b', 1)]
= [(a, 0), (b, 1)] \otimes [(a', 0), (b', 1)]
= \xi(a, b) \otimes \xi(a', b')
\]

For another case of \((a, b) \oplus (a', b')\), we also have
\[
\xi((a, b) \oplus (a', b'))
= \xi(a \lor a', b \lor b')
= [(a \lor a', 0), (b \lor b', 1)]
= [(a, 0) \lor (a', 0), (b, 1) \lor (b', 1)]
= [(a, 0), (b, 1)] \lor [(a', 0), (b', 1)]
= \xi(a, b) \lor \xi(a', b')
\]

Hence the map \(\xi : L_1 \odot L_2 \rightarrow K(L)\) is an embedding, that is,

**Theorem 2.** For every interlaced bilattice \(L_1 \odot L_2\), there exists a lattice \(L\) such that it is embedded into a weak interlaced bilattice \(K(L)\).

From these results, we have a main theorem.

**Theorem 3.** Every interlaced bilattice \(\mathcal{W}\) can be embedded into a weak interlaced bilattice \(K(L)\) for some lattice \(L\).

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References


